EPFL

# Water flow in soils (II)

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# Today’s Lecture

- **Flow in unsaturated soil**
- **Derivation of the governing flow equation** (Richards equation)
- **Variation of hydraulic conductivity in unsaturated soil**
- **Interpretation of hydraulic head profiles in soil**
- **Soil hydraulic properties**

See pages 14-27 of Notes 3.pdf

# Unsaturated flow

In unsaturated media, the water content $\theta$ is less than $\theta_s$ and, in the transient regime, water flow depends on:

- temporal variations in soil water content;
- conservation of mass (water) together with the equation of motion (Darcy).

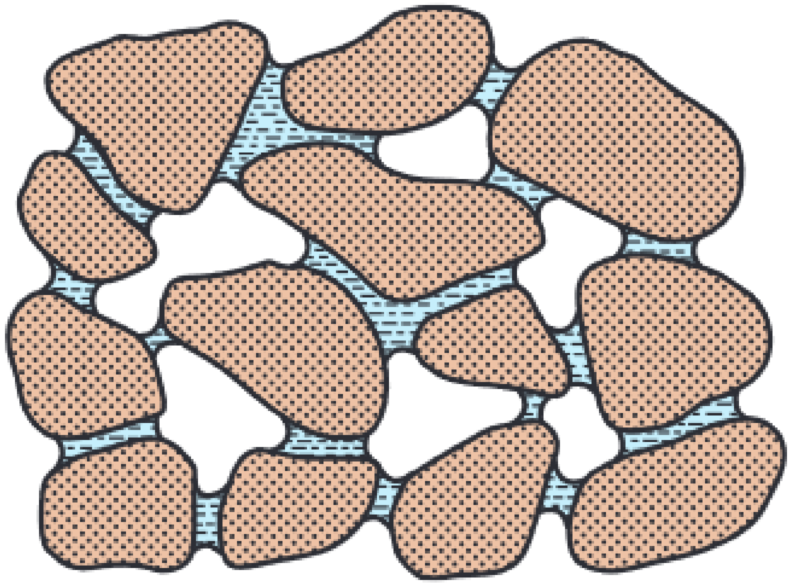


**Fig. 8.1.** Water in an unsaturated coarse-textured soil.

Hillel (2003)

# Generalization of Darcy's law for unsaturated soils

In unsaturated media, the hydraulic conductivity and pressure head depend on water content:

$$\boxed{K = K(\theta) \qquad \text{and} \qquad h = h(\theta)}$$

The Darcy equation becomes: $$q = -K(\theta)\nabla H$$

With $H = h(\theta) + z$: $$q = -K(\theta)\nabla[h(\theta) + z]$$

In 1D (vertical): $$q = -K(\theta)\left(\frac{\partial h(\theta)}{\partial z} + 1\right)$$

# Conservation of mass for unsaturated soils

Darcy's law: $q = -K(\theta)\nabla H$

Continuity equation: $\dfrac{\partial \theta}{\partial t} = -\nabla \cdot q + r_w$

$$\boxed{\frac{\partial \theta}{\partial t} = \nabla \cdot [K(\theta)\nabla(h(\theta) + z)] + r_w}$$

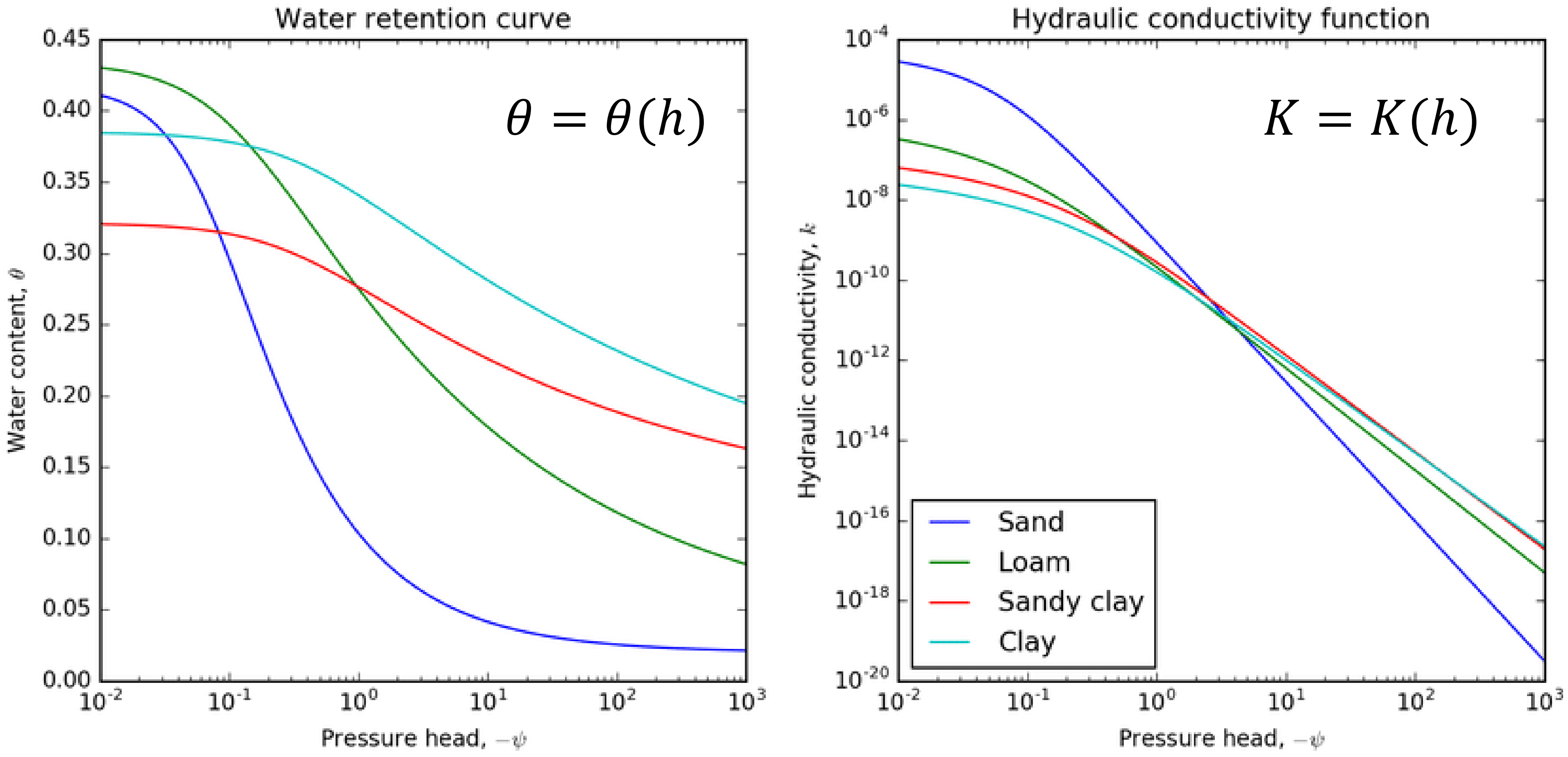
**Unsaturated flow equation**

Note:

- We need to know $K(\theta)$ and $h(\theta)$
- $r_w$= Root Water Uptake (if vegetation is present)
- Two unkowns, $\theta$ and $h$

We need closure equations

# Soil hydraulic functions (example)




Cockett et al. (2017)

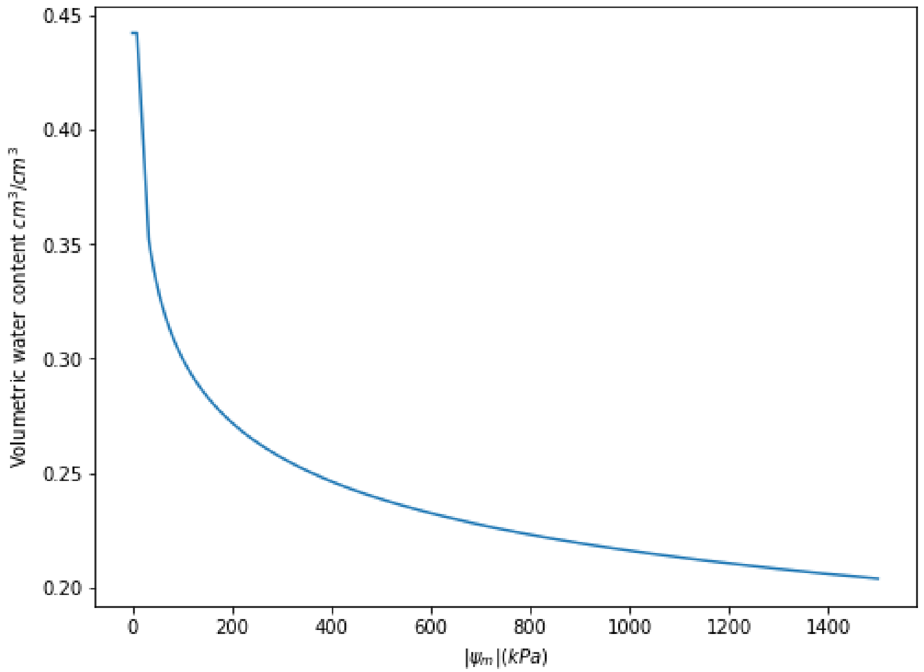
# Soil hydraulic functions (example)

**See computer Lab:** Pedotransfer functions (**Assignment 1 & 2**)

One of the most widely known and practical set of pedotransfer functions was developed by Saxton and Rawls in 1986 and refined in 2006.

```python
# Import modules
import numpy as np
import pandas as pd
```

```python
def ptf(clay, sand, om=0.02, ec=0, rho=np.nan):
    """
    Set of pedotransfer functions to determine soil physical properties from
    soil texture information and organic matter.

    Inputs: clay, sand, and organic matter (om) represented as a fraction. Values range between 0 and 1.

    Outputs: S is a dictionary containing scalar values.
             V is a Pandas dataframe containing vector values across the range from 1500 to 0 kPa.

    """


    # Section numbering does NOT follow the Saxton and Rawls paper
    # 1. Permanent wilting point
    # Calculation of soil water retention at -1500 kPa of tension (Table 1. Eq. 1)
    theta_1500t = -0.024*sand + 0.487*clay + 0.006*om + 0.005*(sand*om) - 0.013*(clay*om) + 0.068*(sand*clay) + 0
    theta_1500 = theta_1500t + (0.14*theta_1500t - 0.02)

    # 2. Field capacity
    # Calculation of soil water retention at -33 kPa of tension (Table 1. Eq. 2)
    theta_33t= -0.251*sand + 0.195*clay + 0.011*om + 0.006*(sand*om) - 0.027*(clay*om) + 0.452*(sand*clay) + 0.29
    theta_33 = theta_33t + 1.283*(theta_33t)**2 - 0.374*(theta_33t) - 0.015

    # 3. Saturation-Field capacity.
    # Volumetric soil water content between 0 and -33 kPa of tension (Table 1. Eq. 3).
    thetaS_33t= 0.278*sand + 0.034*clay + 0.022*om - 0.018*(sand*om) - 0.027*(clay*om) - 0.584*(sand*clay) + 0.07
    thetaS_33= thetaS_33t + (0.636*thetaS_33t - 0.107)
```



# Unsaturated flow equation

The unsatured flow equation can be written in different forms:

- **Head-based formulation**

Apply the chain rule $\frac{\partial\theta}{\partial t} = \frac{\partial\theta}{\partial h}\frac{\partial h}{\partial t}$ and introduce the specific water capacity $C(h) = \frac{\partial\theta}{\partial h}$

$$\Rightarrow \quad C(h)\frac{\partial h}{\partial t} = \nabla\cdot[K(h)\nabla(h+z)] + r_w$$

**Richards equation**

- **Saturation-based formulation**

Apply the chain rule $K(h)\nabla h = K(h)\frac{\partial h}{\partial\theta}\nabla\theta$ and introduce the soil water diffusivity $D(\theta) = \frac{K(\theta)}{C(\theta)} = K(\theta)\frac{\partial h}{\partial\theta}$

$$\Rightarrow \quad \frac{\partial\theta}{\partial t} = \nabla\cdot[D(\theta)\nabla\theta] + \frac{\partial K(\theta)}{\partial z} + r_w$$

**Richards equation** (diffusive form) or **Fokker-Planck equation**

# Solving the Richards equation

**Equation:** $$C(h)\frac{\partial h}{\partial t} = \nabla \cdot [K(h)\nabla(h+z)] + r_w$$

- Highly non-linear partial differential equation (PDE);
- Soil hydraulic functions (c(h) and K(h)) must be known; they can be obtained from the functions $h(\theta)$ and $\mathrm{K}(\theta)$; in the presence of plants, the root extraction function $r_w$ must be formulated mathematically;
- A unique solution requires specification of the system state at the start of the simulation (initial conditions, **IC**) and the conditions prevailing at the system's boundaries during simulation (boundary conditions, BC);
- The solution is generally based on numerical methods (finite differences, finite éléments). Analytical solutions exist only for very restrictive ICs and BCs

**Result:** space-time variations of pressure head $h = h(x, y, z, t)$

# Solving the Richards equation

## Van Genuchten (1980)

$$\theta = \theta_r + (\theta_s - \theta_r)\left[1 + (\alpha\psi)^n\right]^{-m}$$

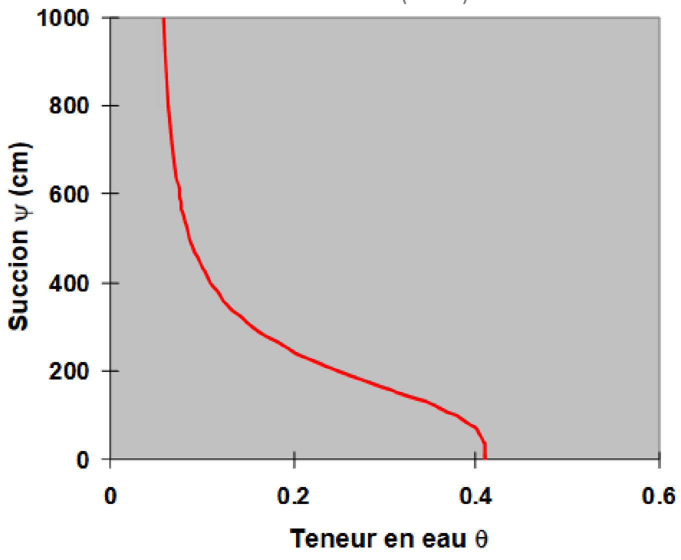
$$\psi = \frac{\left[\left(\dfrac{\theta - \theta_r}{\theta_s - \theta_r}\right)^{-1/m} - 1\right]^{1/n}}{\alpha}$$

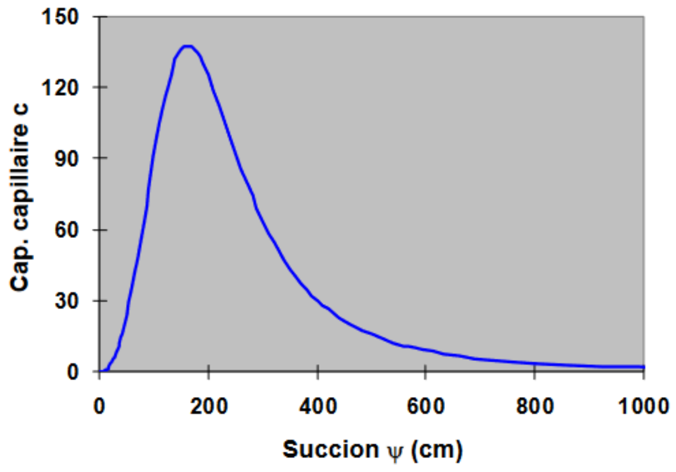
$$c(\psi) = \frac{(\theta_s - \theta_r)(-m)\alpha n(-1)(\psi)^{n-1}}{\left[1 + (\alpha\psi)^n\right]^{1+m}}$$

$\theta_s$ et $\theta_r$ : teneur en eau à saturation et teneur en eau résiduelle

$\alpha$, n et m : constantes

Mermoud (2006)



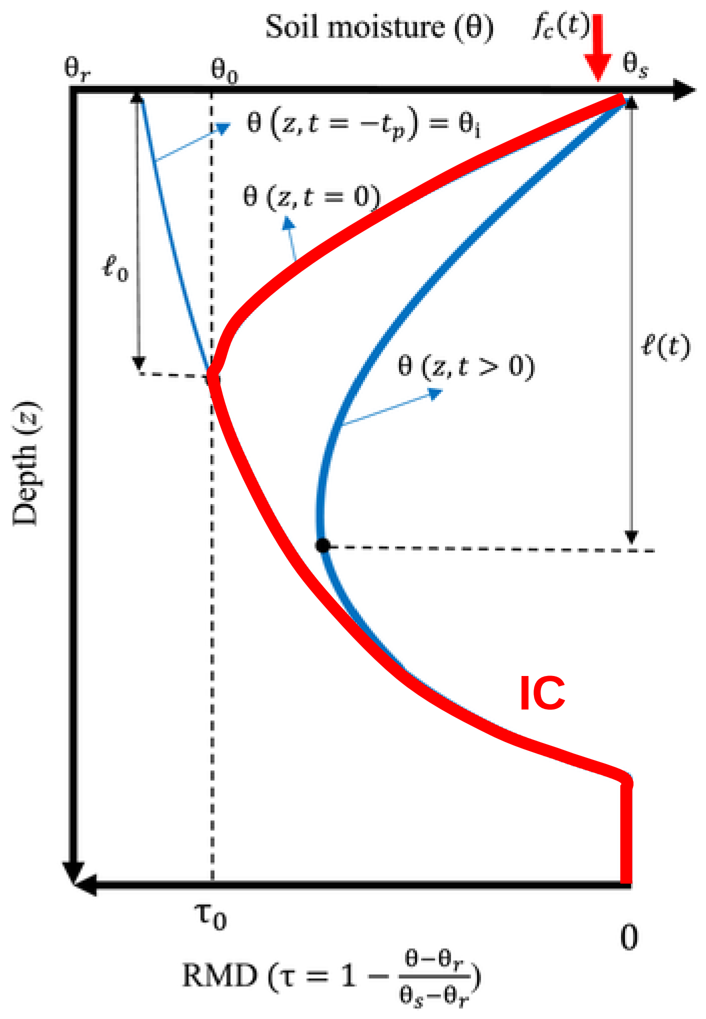
# Solving the Richards equation

### 1D Example

- Specify the **initial condition (IC)**:

  $h(z, t = 0) = h_i(z)$ with $0 \leq z \leq L$

  where $h_i$ is the initial pressure profile, $L$ is the lower limit of the model domain




Hooshyar & Wang (2016)

# Solving the Richards equation

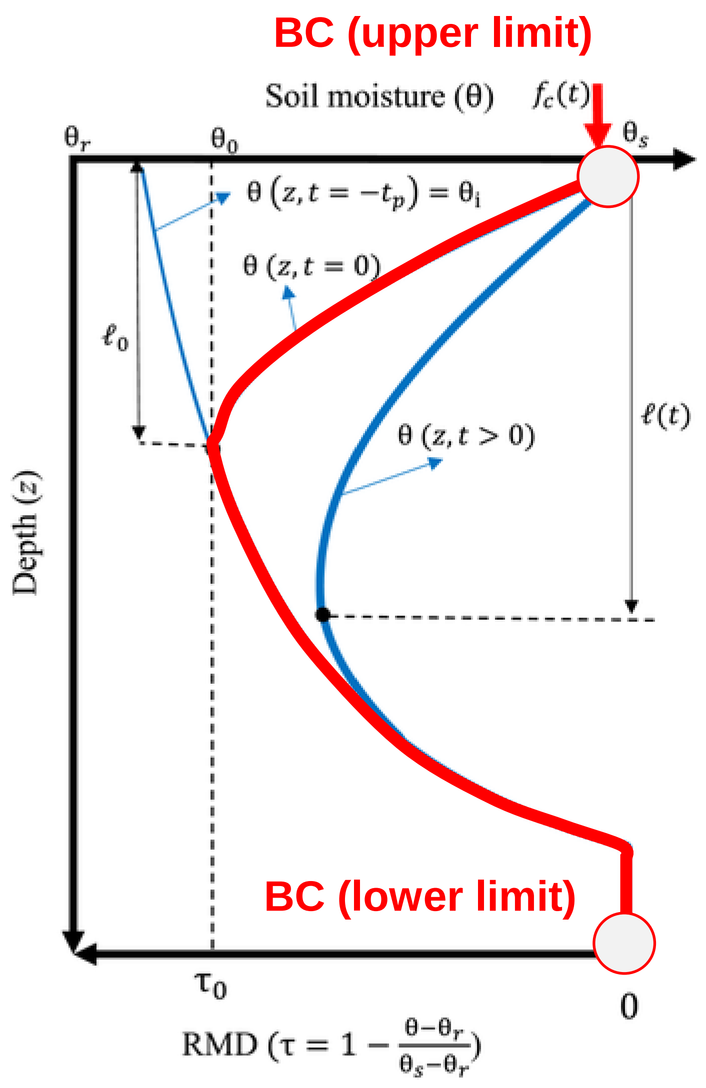
### 1D Example

- Set the <u>boundary conditions (BC)</u>. Two main types:
  - Pressure head is imposed (**Dirichlet**):

$$h(0,t) = h_0(t) \text{ with } t > 0$$
$$h(L,t) = h_L(t) \text{ with } t > 0$$

  - Flux is imposed (**Neumann**)

$$\left(-K\frac{\partial h}{\partial z} + K\right)\Big|_{z=0} = q_0(t)$$
$$\left(-K\frac{\partial h}{\partial z} + K\right)\Big|_{z=L} = q_L(t)$$

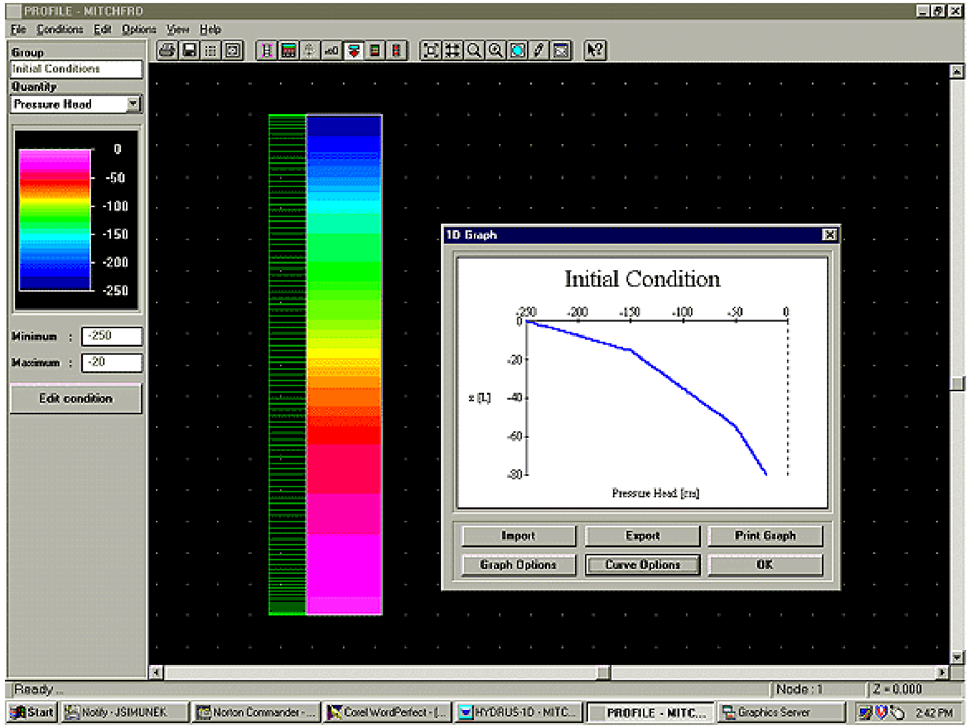



Hooshyar & Wang (2016)

# Solving the Richards equation

The complete mathematical model includes:

- the basic equation;
- knowledge of model parameter values
- initial condition
- boundary conditions
- description of the system geometry (field of study)

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# Solving the Richards equation

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**Example 2-10: A Numerical Approximation to the Richards Equation**

Find: A numerical approximation to changes in water content with time under vertical unsaturated and nonsteady flow into a soil with a known initial water content, and known hydraulic functions $\theta(\psi)$ and $K(\psi)$.

Solution: We first discretize the soil profile (the spatial domain) into units of $\Delta z$, and the time into units of $\Delta t$. We denote the water content at the i-th depth increment and at the j-th time step as $\theta_i^j$; similarly we express the corresponding matric head as $h_i^j$. The hydraulic conductivity, which is a function of the matric head (or of the water content), relates transfer between two soil layers and thus must be averaged to represent both layers as: $K([h_i^j + h_{i+1}^j]/2) = K_{i+1/2}^j$. The resulting numerical approximation to Eq. (92) for vertical flow is then given by

$$\frac{\partial \theta}{\partial t} = \frac{\theta_i^{j+1} - \theta_i^j}{\Delta t} = \frac{\partial}{\partial z}\left[K(h)\left(\frac{\partial h}{\partial z} + 1\right)\right] = \frac{(h_{i-1}^j - h_i^j + \Delta z)K_{i-1/2}^j}{\Delta z^2} - \frac{(h_i^j - h_{i+1}^j + \Delta z)K_{i+1/2}^j}{\Delta z^2} \quad \textbf{(101)}$$

Combining terms and rearranging allows us to solve for the only unknown in Eq.101, which is $\theta_i^{j+1}$, the water content for the i-th depth increment at the next (future) time step:

$$\theta_i^{j+1} = \theta_i^j + \frac{\Delta t}{\Delta z^2}\left[(h_{i-1}^j - h_i^j + \Delta z)K_{i-1/2}^j - (h_i^j - h_{i+1}^j + \Delta z)K_{i+1/2}^j\right] \quad \textbf{(102)}$$

Because this is a discrete approximation of a continuous process, increments $\Delta z$ and $\Delta t$ should be kept small; the actual selection criteria are beyond the scope of this course. In addition, initial soil water contents with respect to depth and the conditions at the boundaries of the flow domain need to be specified.

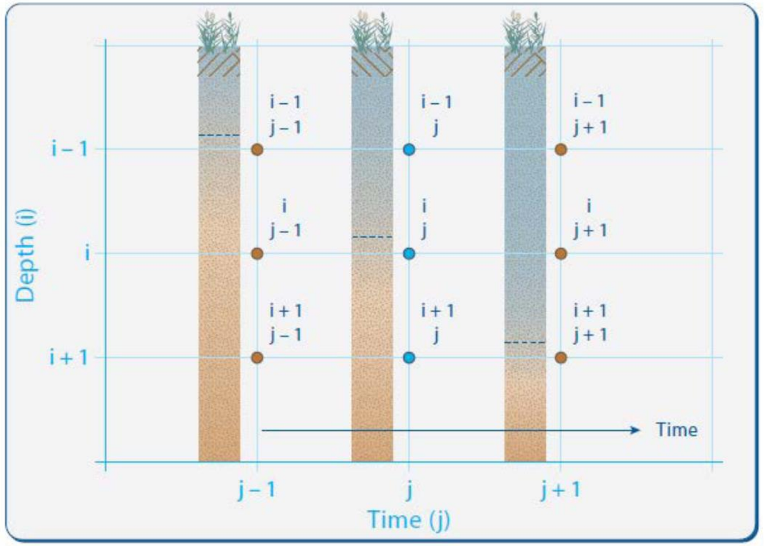


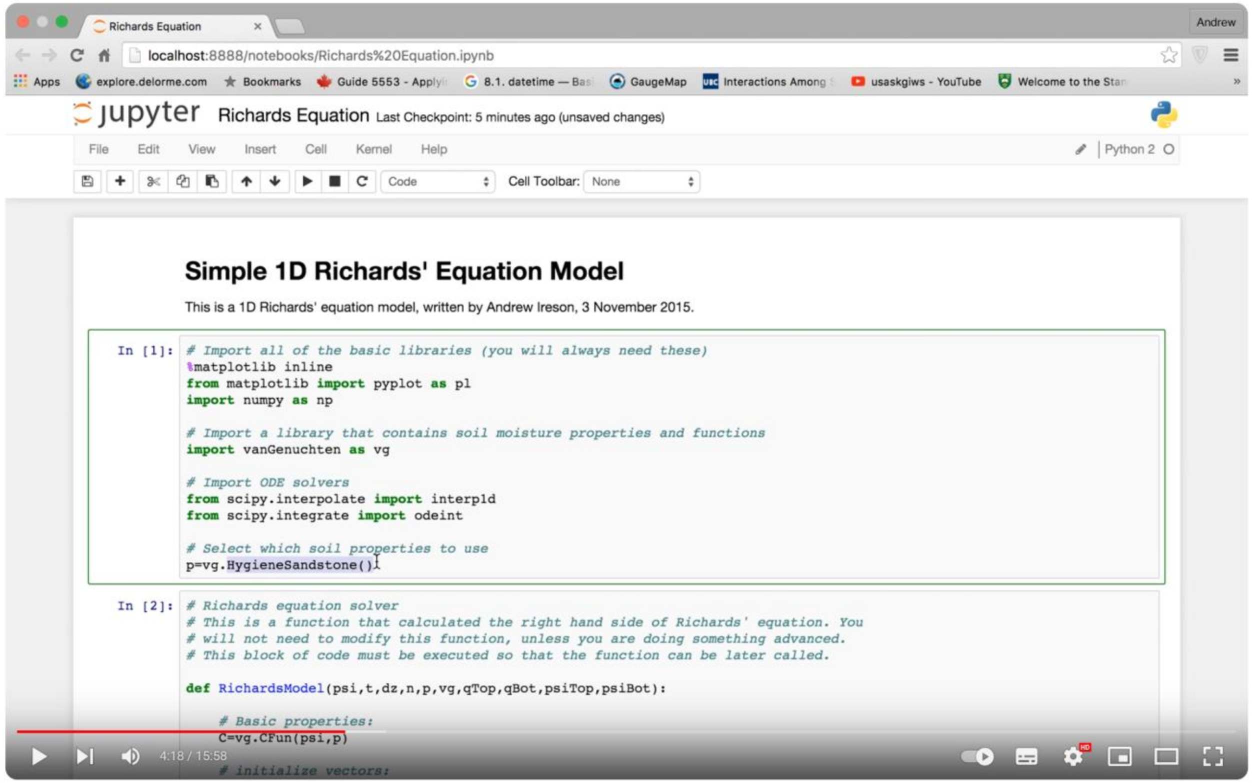

Fig.2-15: Diagram of space-time grid for 1-D finite difference numerical approximation.

Source: Or, Tuller, & Wraith, 1994-2018

(Optional) Self-Study

# Solving the Richards equation

**See computer Lab:** 1D Richards Equation (**Assignment 3**)



# Solving the Richards equation

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https://doi.org/10.5194/gmd-15-75-2022


Geoscientific Model Development
Open Access

## Implementing the Water, HEat and Transport model in GEOframe (WHETGEO-1D v.1.0): algorithms, informatics, design patterns, open science features, and 1D deployment

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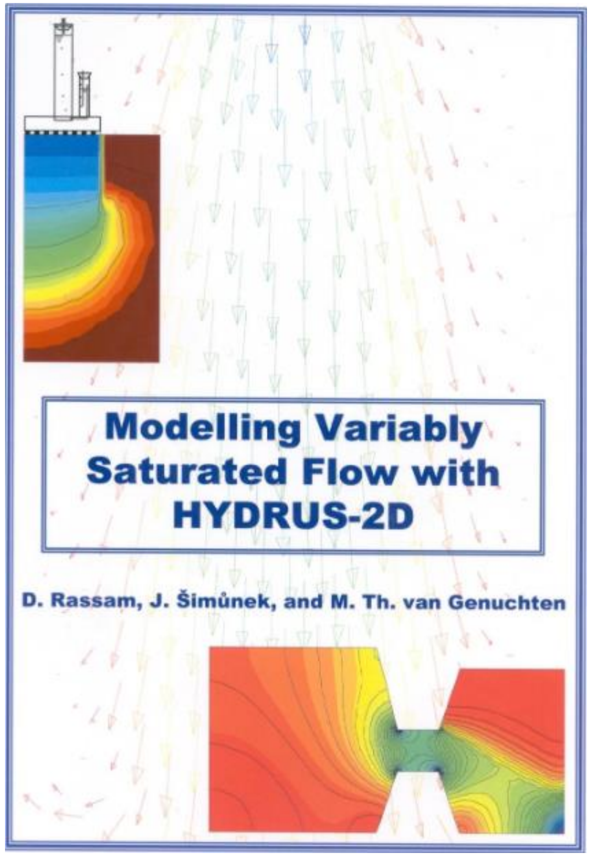
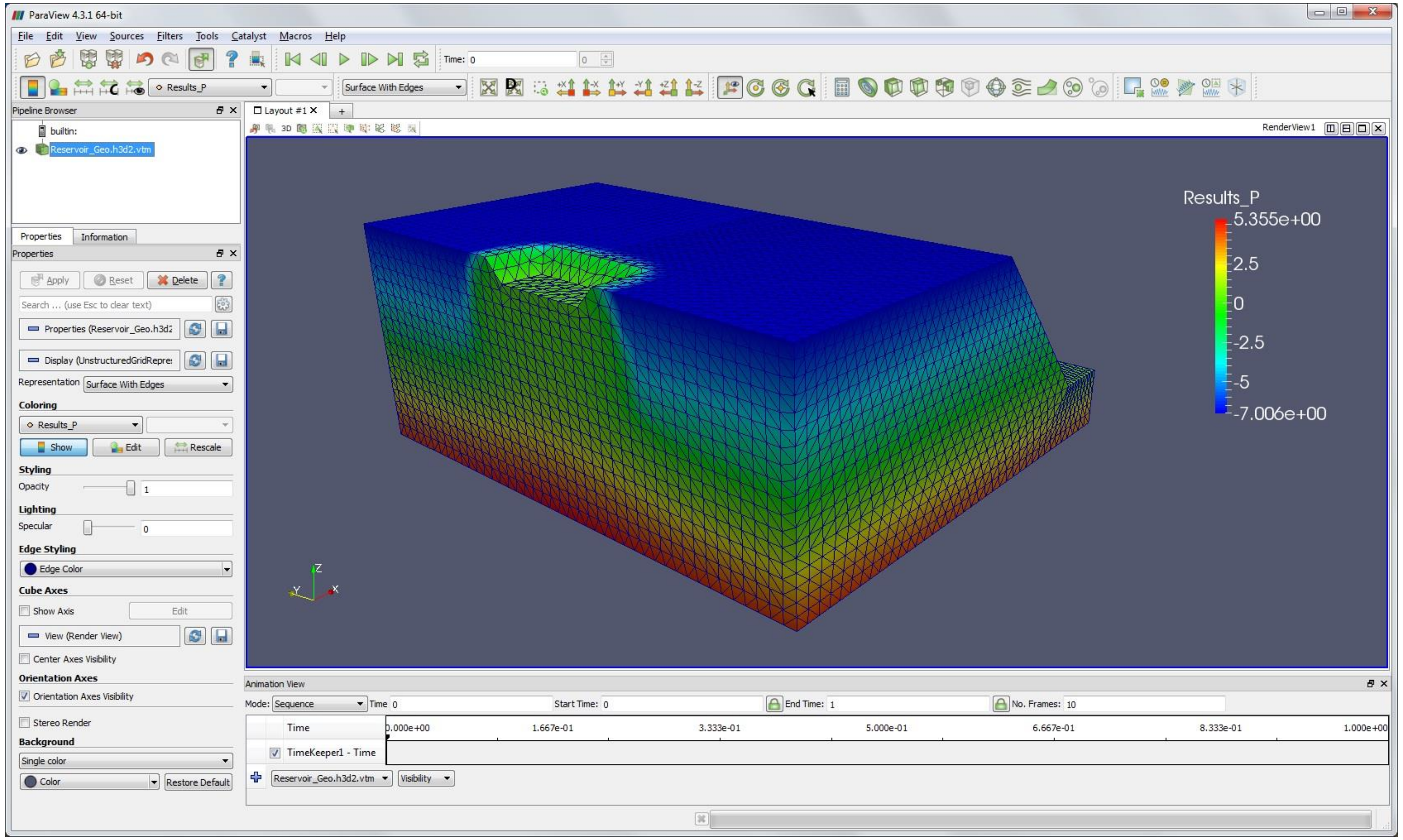
https://github.com/GEOframeOMSProjects/OMS_Project_WHETGEO1D

# Solving the Richards equation

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# Richards equation: example application

Richards' equation can be used to predict the spatio-temporal evolution of the water content or pressure load of the soil under the effect of irrigation.

**Example:** Irrigation of 4 mm/h for 2 h

IC: pressure head at start of simulation

BC:

- Top: q = 4mm/h for z=0 and 0 <t < 2h
- Bottom: depends on local conditions (e.g., watertable)

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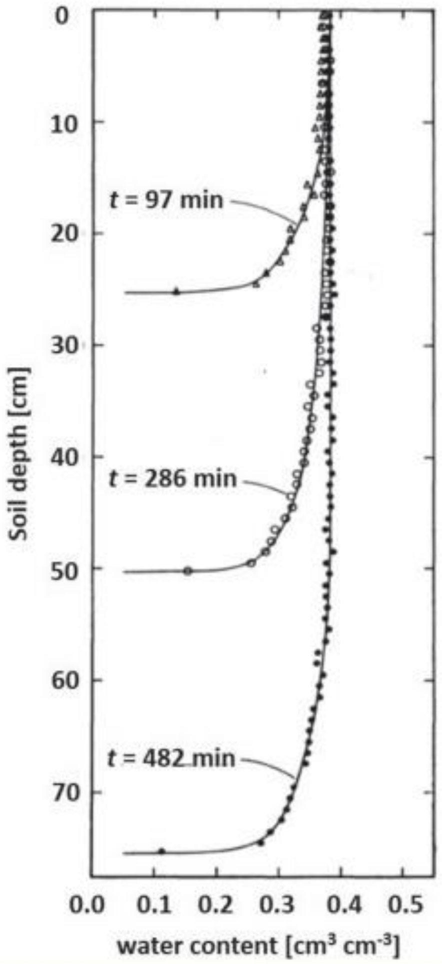



Fig. 4. Infiltration into Hesperia sandy loam soil (initially air dry) showing the soil water content with depth and the propagation of the wetting front with time during infiltration. (Reprinted with permission from Davidson et al., 1963.)

Vereecken et al. (2019), VZJ: link here

# Experimental study of soil-atmosphere water transfer

## Experimental setup

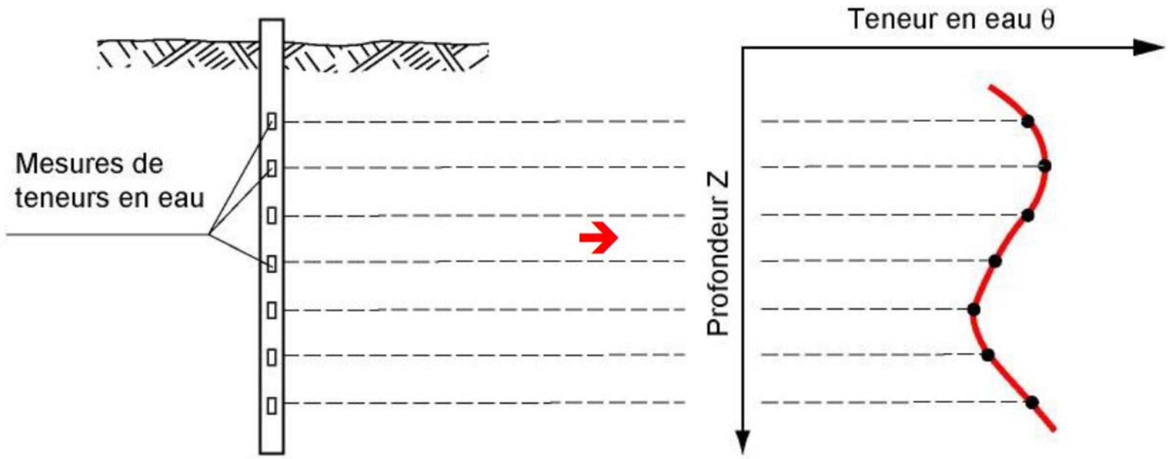
- Measurements: space-time variation of water content and pressure hear along the soil profile
- Experimental setup:
  - Equipment to monitor the soil water content (e.g., TDR)
  - Tensiometer
- Hypotheses:
  - Vertical flow only (1D)
  - No root water uptake
  - Homogeneous soil
- Interpretation of measurements:
  - Continuity equation
  - Darcy's law

$$\frac{\partial \theta}{\partial t} = -\frac{\partial q}{\partial z} \quad \text{with} \quad q = -K(\theta)\frac{\partial H}{\partial z}$$

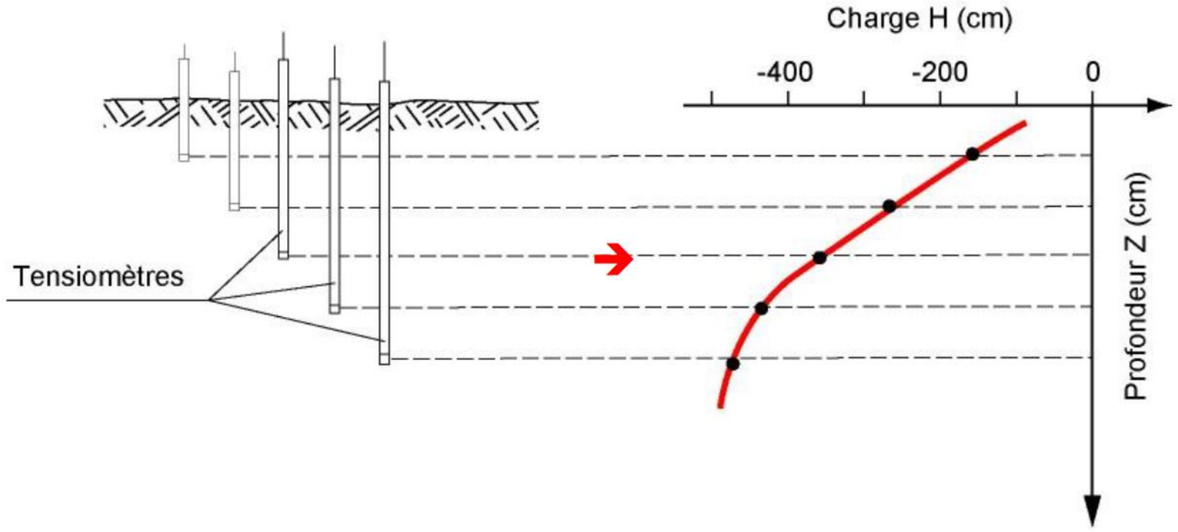
- Results:
  - Flow direction
  - Average flux between two measurements
  - Instantaneous flux

# Experimental study of soil-atmosphere water transfer

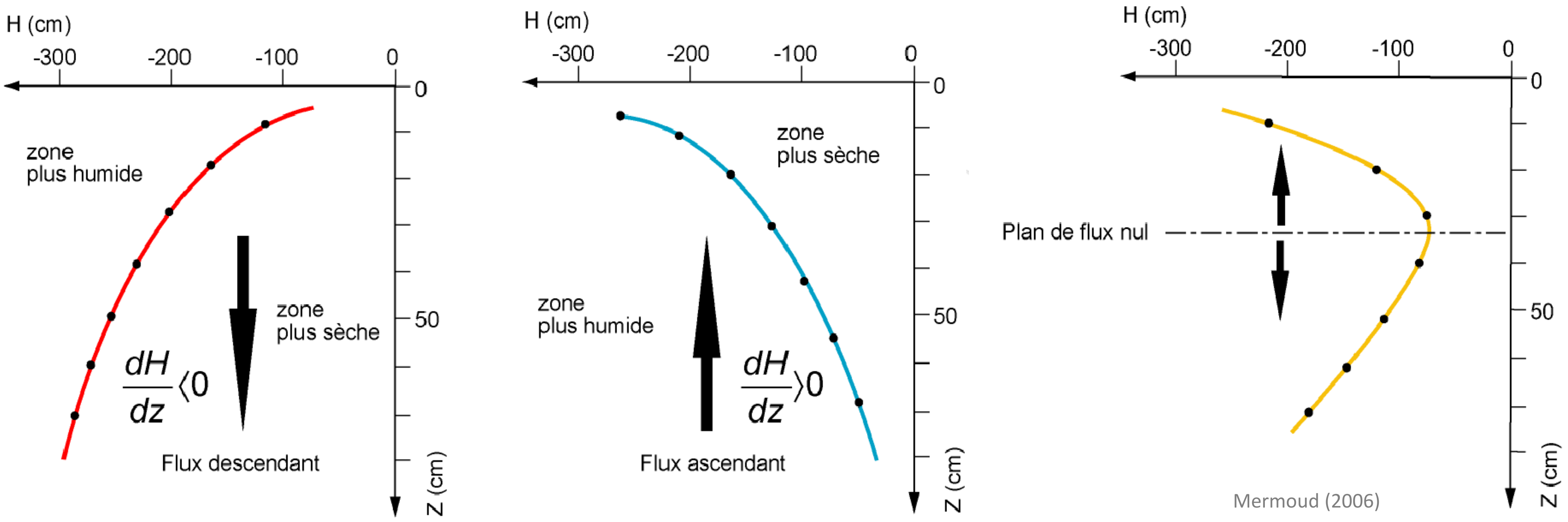
**Water content measurements**




**Water potential measurements**



# Experimental study of soil-atmosphere water transfer




**Estimating flow direction from the sign of the water potential gradient**

# Experimental study of soil-atmosphere water transfer

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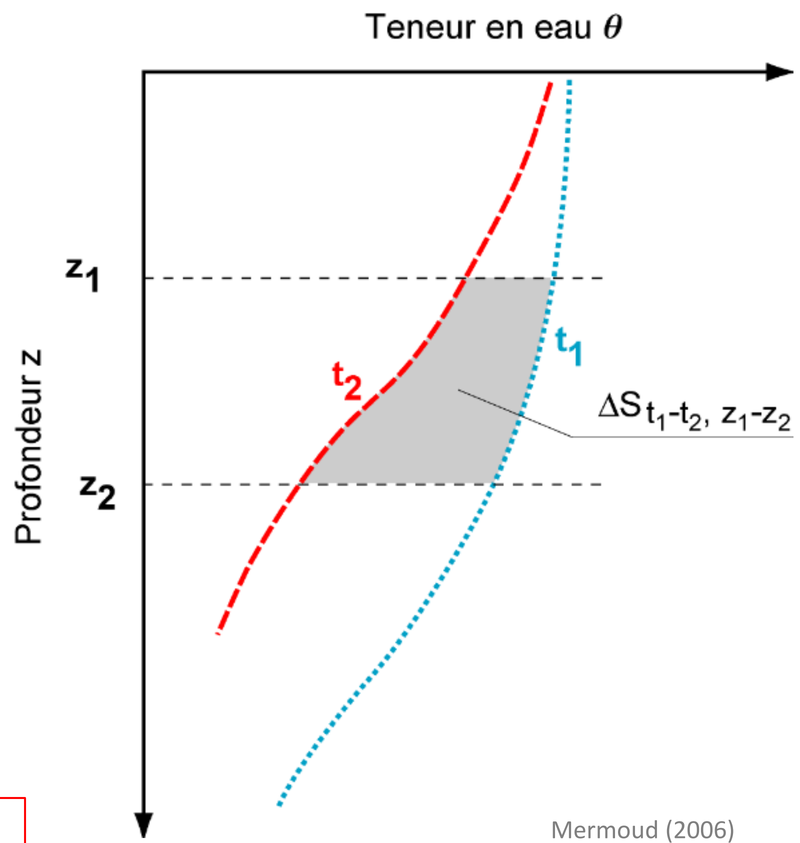
## Estimating the flux using the continuity equation

Integrating at time *t* between depts $z_1$ and $z_2$, we obtain:

$$\int_{z_1}^{z_2} \frac{\partial \theta}{\partial t} dz = -\int_{z_1}^{z_2} \frac{\partial q}{\partial z} dz$$

$$\underbrace{\frac{\partial}{\partial t} \int_{z_1}^{z_2} \theta \cdot dz}_{\text{Temporal variation of the water stock S between } z_1 \text{ and } z_2} = -q\Big|_{z_1}^{z_2} = \underbrace{q(z_1) - q(z_2)}_{\text{Flux difference between } z_1 \text{ and } z_2}$$

$$\Rightarrow \quad \frac{\partial S_{z_1 - z_2}}{\partial t} = q(z_1) - q(z_2) \quad \Rightarrow \quad \boxed{q(z_2) = q(z_1) - \frac{\Delta S_{z_1 - z_2}}{\Delta t}}$$




Mermoud (2006)

To estimate the flux at depth $z_2$, you need to know the flux at $z_1$ and the variation of water volume occurring over the time $\Delta t$ between the two depths

# Experimental study of soil-atmosphere water transfer

**Experimental estimation of the flux**

$$\boxed{q(z_2) = q(z_1) - \frac{\Delta S_{z_1 - z_2}}{\Delta t}}$$

**Special cases:**

- No-flow at the surface (e.g. impermeable surface):
  - $z_1 = 0$
  - $q(z_1) = 0$

  $\longrightarrow$ $$q(z) = -\frac{\Delta S_{0-z}}{\Delta t}$$
- Flux *i* imposed at surface (e.g., controlled infiltration experiment):
  - $Z_1 = 0$
  - $q(z_1) = i$

  $\longrightarrow$ $$q(z) = i - \frac{\Delta S_{0-z}}{\Delta t}$$
- No-flow at known depth $z_o$ (e.g., as detected from tensiometers)
  - $z_1 = z_0$
  - $q(z_0) = 0$

  $\longrightarrow$ $$q(z) = -\frac{\Delta S_{z_0-z}}{\Delta t}$$

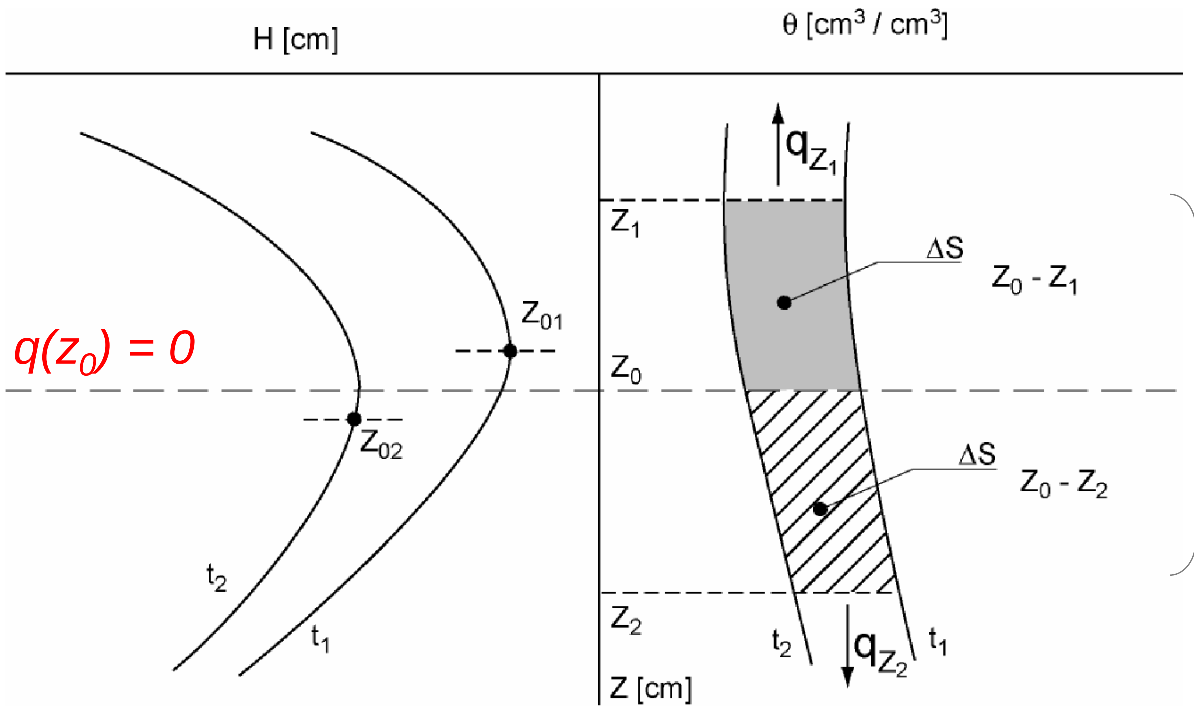
next slide

**Rainfall simulator**
(controlled infiltration experiment)

Strauss & Hosl (2016)

# Experimental study of soil-atmosphere water transfer

## Experimental estimation of the flux (presence of a no-flow depth)



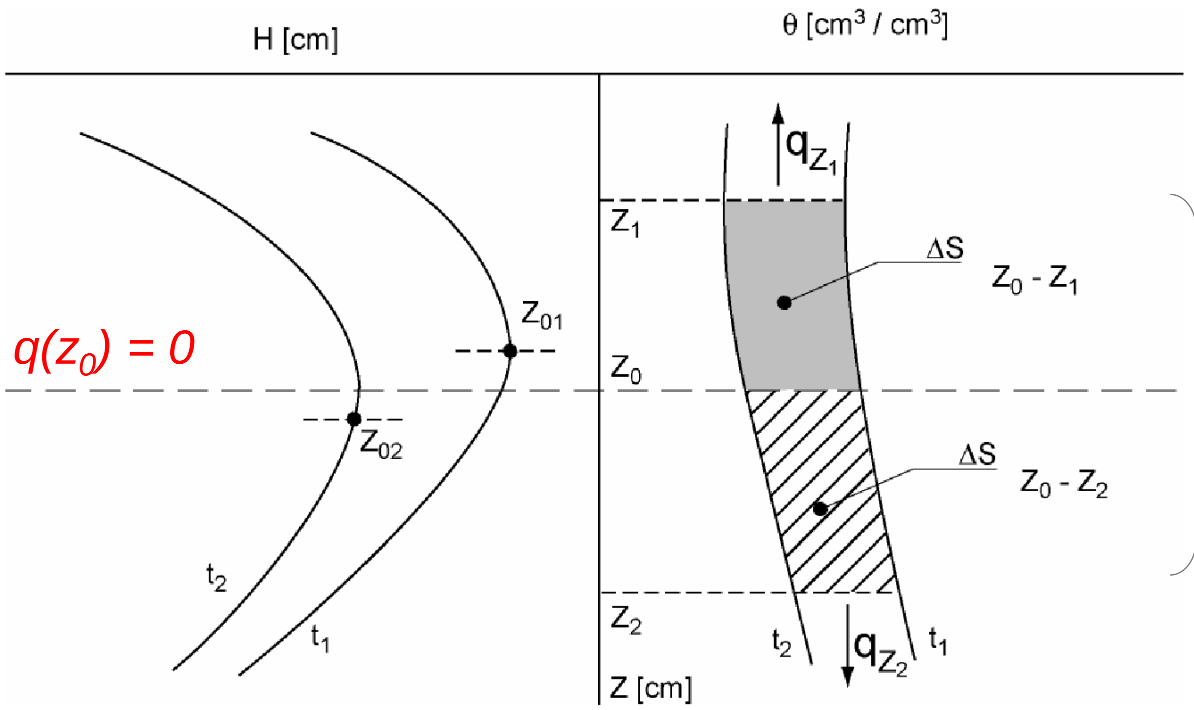

**Profils de charge** **Profils hydriques**

$$q(z_1) = +\frac{\Delta S_{z_0 - z_1}}{\Delta t}$$

$$q(z_2) = -\frac{\Delta S_{z_0 - z_2}}{\Delta t}$$

# Experimental study of soil-atmosphere water transfer

## Experimental estimation of the flux (presence of a no-flow depth)




**Profils de charge** **Profils hydriques**

Evaporation

**Surface** $(z = 0)$

$$q(z_1) = +\frac{\Delta S_{z_0 - z_1}}{\Delta t}$$

$$q(z_2) = -\frac{\Delta S_{z_0 - z_2}}{\Delta t}$$

$$q(z = 0) = \frac{\Delta S_{z_0 - 0}}{\Delta t}$$

$$q(z = D) = -\frac{\Delta S_{z_0 - D}}{\Delta t}$$

**Water table** $(z = D)$

Groundwater recharge

# Experimental study of soil-atmosphere water transfer

## Estimation of the instantaneous flux

Necessary information:

- Hydraulic conductivity function
- Water content (or pressure head) at depth *z* and time *t*
- Water potential gradient at depth *z* and time *t* (this can be measured with two tensiometers right above and below *z*)

$$\boxed{q(z,t) = -K(\theta(z,t))\frac{\partial H(z,t)}{\partial z}}$$

# This week exercises & assignments

- **Exercises** for **Weeks 5** are available in Moodle
  - In case you want to try more problems, additional exercises are also available (just try the problem before checking the solution!)
- **Simplified capillary model of a soil** (optional)
  - See file in Moodle – it contains some questions designed to show how the capillary tube model relates to a soil.
- **Computer Lab:** continuation of Assignment 3 (Q&A) + optional assignment (soil water balance)
- **For next week:** read Notes 4.pdf (final set of notes).